MATH 105 101 Midterm 1 Sample 3

1. (15 marks)

(a) (3 marks) Let

$$f(x, y) = y^2 + y \ln x.$$

Compute both first-order partial derivatives of f at the point $(1, 2)$. *Simplify your answers.*

Solution: Calculate the first-order partial derivatives of f:

$$f_x(x, y) = y/x, \qquad f_y(x, y) = 2y + \ln x.$$

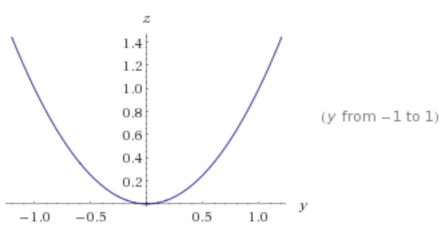
Evaluate f_x and f_y at $(1, 2)$:

$$f_x(1, 2) = 2/1 = 2, \qquad f_y(1, 2) = 2(2) + \ln 1 = 4.$$

Thus, $f_x(1, 2) = 2$, and $f_y(1, 2) = 4$.

(b) (2 marks) Given $f(x, y)$ as in part (a), sketch the trace of the surface $z = f(x, y)$ in the $x = 1$ plane.

Solution: In the $x = 1$ plane, the trace of the surface is given by the equation $z = y^2 + y \ln 1 = y^2$, which is a parabola in the yz-plane.



(c) (2 marks) Find a unit vector parallel to $\langle -2, 1, 2\rangle$.

Solution: Let $\mathbf{v}$ be such a vector. Then, since $\mathbf{v}$ is parallel to $\langle -2, 1, 2\rangle$, we have that $\mathbf{v} = c\langle -2, 1, 2\rangle = \langle -2c, 1, 2c\rangle$ for some real constant c. Since $\mathbf{v}$ is a unit vector, which means it has length 1, we get:

$$\begin{aligned} 1 &= \sqrt{(-2c)^2 + c^2 + (2c)^2} = \sqrt{9c^2} = 3|c| \\ 1/3 &= |c| \\ \Rightarrow c &= \pm 1/3. \end{aligned}$$

Thus, a unit vector in $\mathbb{R}^3$ which is parallel to the vector $\langle -2, 1, 2\rangle$ can either be $\langle -2/3, 1/3, 2/3\rangle$ and $\langle 2/3, -1/3, -2/3\rangle$.

(d) (2 marks) Find an equation for the plane passing through the point $P(1, 2, 3)$ that is orthogonal to the vector $\langle 4, 0, -1\rangle$.

Solution: An equation of the plane passing through the point $P(1, 2, 3)$ that is orthogonal to the vector $\langle 4, 0, -1\rangle$ is:

$$\begin{aligned} 4(x-1) + 0(y-2) - 1(z-3) &= 0 \\ \Rightarrow 4x - z &= 1. \end{aligned}$$

(e) (3 marks) Determine if the plane described by the equation:

$$2x - 5y + 2z = -1,$$

is orthogonal to the plane given in part (d).

Solution: To determine if the plane $2x - 5y + 2z = -1$ is orthogonal to the plane obtained in part (d), we want to determine if their normal vectors are orthogonal to each other, that is, whether $\langle 2, -5, 2\rangle$ is orthogonal to $\langle 4, 0, -1\rangle$. Since:

$$2(4) + (-5)(0) + 2(-1) = 6 \neq 0,$$

$\langle 2, -5, 2\rangle$ is not orthogonal to $\langle 4, 0, -1\rangle$. Thus, the planes are not orthogonal to each other.

(f) (3 marks) Assume that $f(x, y)$ has continuous partial derivatives of all orders. If

$$f_y(x, y) = x^3 + 2x^2y,$$

compute f_{xyx}. State in detail any result that you use.

Solution: Since $f(x, y)$ has continuous partial derivatives of all orders, in particular, f_{xy} and f_{yx} are continuous. By Clairaut's Theorem, that means $f_{xy} = f_{yx}$. Thus, $f_{xyx} = (f_{xy})_x = (f_{yx})_x = f_{yxx}$. Hence, to find f_{xyx}, we have:

$$f_{yx}(x, y) = 3x^2 + 4xy, \qquad f_{yxx} = 6x + 4y.$$

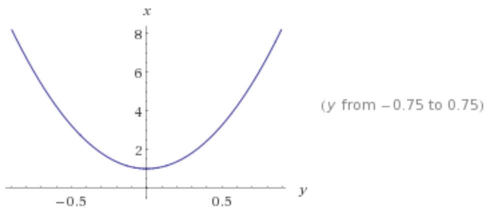
Thus, $f_{xyx} = 6x + 4y$.

2. (5 marks) Consider the surface S given by:

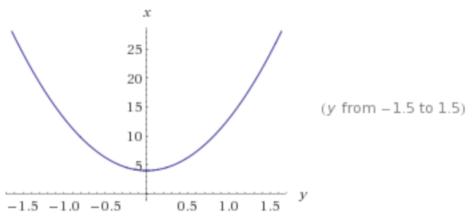
$$z^2 = x - 9y^2.$$

(a) (4 marks) Find and sketch the level curves of S for $z_0 = 1$ and $z_0 = 2$.

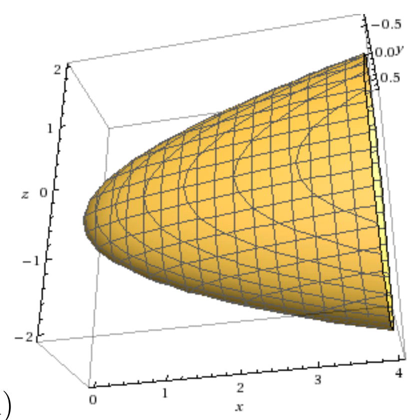
Solution: For $z_0 = 1$, the level curve is $1 = x - 9y^2$, which is, $x = 9y^2 + 1$ (a parabola in the xy-plane).



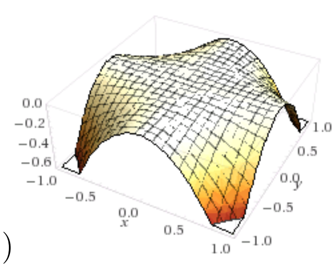
For $z_0 = 2$, the level curve is $4 = x - 9y^2$, which is, $x = 9y^2 + 4$ (a parabola in the xy-plane).



(b) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?



(A)



(B)

Solution: The answer is (A), since in (B), the level curves at $z_0 = 1$ and $z_0 = 2$ seem to be empty, not parabolas.

3. (10 marks) Let R be the semicircular region $\{x^2 + y^2 \leq 4, x \geq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 - 2x + y^2.$$

on the *boundary of the region R*.

Solution: The boundary of the region R consists of two pieces: the semicircular arc which can be parametrized by $x = 2\cos\theta$ and $y = 2\sin\theta$ for $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and the vertical segment $x = 0$ for $-2 \leq y \leq 2$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the semicircular arc*: We have that $f(x, y) = g(\theta) = (2\cos\theta)^2 + (2\sin\theta)^2 - 2(2\cos\theta) = 4 - 4\cos\theta$ for $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then, $g'(\theta) = 4\sin\theta = 0$ if and only if $\theta = 0$. So, there are 3 points where extrema can occur: $(2, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).
- *On the vertical segment*: We have that $f(0, y) = h(y) = y^2$ for $-2 \leq y \leq 2$. So, $h'(y) = 2y = 0$ when $y = 0$. So, there are 3 points where extrema can occur: $(0, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).

Evaluate f at those points, we get:

$$f(2, 0) = 0, \quad f(0, -2) = 4, \quad f(0, 2) = 4, \quad f(0, 0) = 0$$

Thus, on the boundary of R, f attains the absolute maximum value 4 at the points $(0, -2)$ and $(0, 2)$ and the absolute minimum value 0 at the points $(2, 0)$ and $(0, 0)$.

4. (10 marks) Find *all* critical points of the following function:

$$f(x, y) = xy - \frac{x^2}{2} - \frac{y^3}{3} + 5.$$

Classify each point as a local minimum, local maximum, or saddle point.

Solution: Compute the first-order partial derivatives of f:

$$f_x(x, y) = y - x \qquad f_y(x, y) \qquad\qquad = x - y^2.$$

Since both f_x and f_y are defined at every point in $\mathbb{R}^2$, the only critical points of f are those at which $f_x = f_y = 0$. If $f_x = 0$, then $x = y$. Replacing $x = y$ into $f_y = 0$, we get:

$$y - y^2 = y(1 - y) = 0 \Rightarrow y = 0, 1.$$

So, we get two critical points $(0, 0)$ and $(1, 1)$. Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = -1, \quad f_{yy} = -2y, \quad f_{xy} = 1, \quad D(x, y) = 2y - 1$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(0,0)$, $D(0,0) = -1 < 0$, so $(0,0)$ is a saddle point.
- At the point $(1,1)$, $D(1,1) = 1 > 0$ and $f_{xx}(1,1) = -1 < 0$, so $(1,1)$ is a local maximum.

5. (10 marks) A firm produces:

$$P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$$

units of goods per week, utilizing x units of labour and y units of capital. If labour costs \$27 per unit, and capital costs \$0.5 per unit, use the method of Lagrange multiplier to find the most cost-efficient division of labour and capital that the firm can adopt if its goal is to produce 6 units of goods per week.

Clearly state the objective function and the constraint. *You are not required to justify that the solution you obtained is the absolute maximum.* **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

Solution: Since labour costs \$27 per unit, and capital costs \$0.5 per unit, the cost function is $C(x,y) = 27x + 0.5y$. The objective function to minimize is the cost function $C(x,y) = 27x + 0.5y$, and the constraint function is $g(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}} - 6 = 0$ (since its goal is to produce 6 units). Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned} \nabla C(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= 0 \end{aligned}$$

More explicitly, we need to solve:

$$\begin{aligned} 27 &= \frac{2}{3}\lambda x^{\frac{-1}{3}}y^{\frac{1}{3}} \\ 0.5 &= \frac{1}{3}\lambda x^{\frac{2}{3}}y^{\frac{-2}{3}} \\ x^{\frac{2}{3}}y^{\frac{1}{3}} - 6 &= 0. \end{aligned}$$

Isolate for λ in the first two equations, we get:

$$\lambda = \frac{81}{2}x^{\frac{1}{3}}y^{\frac{-1}{3}}, \qquad \lambda = \frac{3}{2}x^{\frac{-2}{3}}y^{\frac{2}{3}}.$$

Equate the above equations, we get:

$$\begin{aligned} \frac{81}{2}x^{\frac{1}{3}}y^{\frac{-1}{3}} &= \frac{3}{2}x^{\frac{-2}{3}}y^{\frac{2}{3}} \\ 27x &= y. \end{aligned}$$

Replace $27x = y$ in the third equation, we get $3x - 6 = 0$, so $x = 2$. Thus, $y = 54$ and $\lambda = 27/2$. Therefore, if the firm uses 2 units of labour and 54 units of capital, then it can produce 6 units of good while minimizing its costs.